Mathematics intro

Probabilities, Conditional Independence and Assumption Parlance

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Marginal, Joint and Conditional probabilities

Setup

Probability statements about *random events* A and B

- A: patient dies ($A = 1$)
- B: patient has cancer ($B = 1$)

Say we have 100 patients, we can tabulate them according to their cancer status and whether they died or not.

joint probability table

		A		
		dies	lives	
B	has cancer	5	5	10
	has no cancer	10	80	90
		15	85	100

Marginal probabilities

- *Marginal* probabilities concern probabilities of *one* random event, regardless of the other random event.
- We read these probabilities from the *margins* of the joint probability table.

statement	interpretation
$P(A = 1)$	*marginal* probability that event A occurs
$P(B = 1)$	*marginal* probability that event B occurs

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$P(A = 1) = 15/100$

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		A		
		dies	lives	
B	has cancer			10
	has no cancer			90
				100

$P(B = 1) = 10/100$

Joint Probabilities

- A *joint* probability concerns the probability of two random events *jointly* occurring together.
- These are a based on a single cell in the joint probability table

statement	interpretation
$P(A)$	*marginal* probability that event A occurs
$P(A = 1, B = 1)$	*joint* probability of A and B

		A		
		dies	lives	
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		A		
		dies	lives	
B	has cancer	5		
	has no cancer			
				100

$P(B = 1, A = 1) = 5/100$

Conditional probabilities

- *Conditional* probabilities concern the probability of one random event given that another random event has occurred.
- e.g. *what is the probability that a patient dies ($A = 1$) given that they have cancer ($B = 1$)?*
- These are read from the joint probability table by looking in the row or column of the conditioning event.

statement	interpretation
$P(A)$	*marginal* probability that event A occurs
$P(A, B)$	*joint* probability of A and B
$P(A = 1 \mid B = 1)$	*conditional* probability of A given B

		A		
		dies	lives	
B	has cancer	5	5	10
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		15	85	100

- *marginal* $P(A = 1) = 15/100$

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	has no cancer			

- *marginal* $P(A = 1) = 15/100$

- *conditional* $P(A = 1 \mid B = 1) = 5/10$

Probability rules and identities

Sum rule

A *marginal* probability can be computed by summing over the *joint probabilities* of all possible values of the other random event.

statement	interpretation
$P(A) = \sum_b P(A, B = b)$	marginal is sum over joint

		A	
		dies	lives
B	has cancer	5	
	has no cancer	10	
		15	100

$$\begin{aligned} P(A = 1) &= P(A = 1, B = 1) + P(A = 1, B = 0) \\ &= 5/100 + 10/100 \\ &= 15/100 \end{aligned}$$

Product rule

A *joint* probability can be computed by multiplying the *conditional probability* of one random event given the other random event with the *marginal probability* of the other random event.

statement	**interpretation**
$P(A) = \sum_b P(A, B = b)$	marginal is sum over joint
$P(A, B) = P(A \mid B)P(B)$	product rule

		A		
		dies	lives	
B	has cancer	5		10
	has no cancer			
				100

$$
\begin{aligned}
P(A = 1, B = 1) &= P(A = 1 \mid B = 1)P(B = 1) \\
&= 5/10 * 10/100 \\
&= 5/100
\end{aligned}
$$

With these two rules and basic algebra, we can derive more identities

statement	**interpretation**
$P(A) = \sum_b P(A, B = b)$	marginal is sum over joint
$P(A, B) = P(A \mid B)P(B)$	product rule

Product rule - different form

a *conditional* probability can be computed by dividing the *joint probability* of the two random events by the *marginal probability* of the other random event, since[1]

$$x = y * z \implies y = \frac{x}{z}$$

statement	interpretation
$P(A) = \sum_b P(A, B = b)$	marginal is sum over joint
$P(A, B) = P(A\|B)P(B)$	product rule
$P(A\|B) = \frac{P(A,B)}{P(B)}$	conditional is joint over marginal (follows from product rule)

1. $z \neq 0$

Law of total probability

statement	interpretation
$P(A) = \sum_b P(A, B = b)$	marginal is sum over joint
$P(A, B) = P(A \mid B)P(B)$	product rule
$P(A \mid B) = \frac{P(A,B)}{P(B)}$	conditional is joint over marginal (follows from product rule)
$P(A \mid C) = \sum_b P(A \mid B = b, C)P(B = b \mid C)$	total probability (consequence of marginal vs joint and product rule)

- this identity can be proven quite easily using the product rule and the sum rule Section 5.1

Marginal independence and conditional independence

Marginal independence

statement	interpretation
$P(A, B) = P(A)P(B)$	(marginal) independence of A and B

- knowing A has no information on what to expect of B
- If I roll a die, the result of that die (A) has no information on the weather in the Netherlands (B)

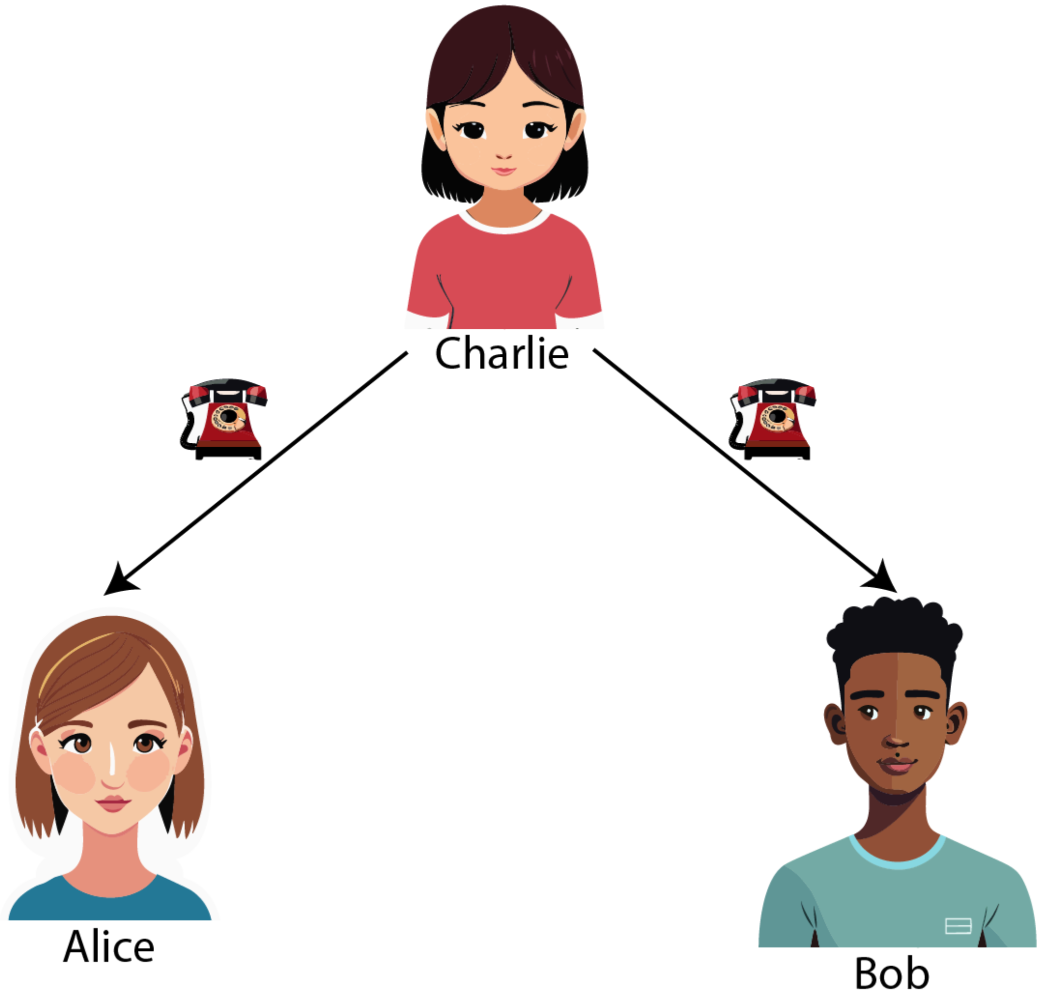
Conditional independence

- some events may not be independent in general, but they may be independent given some other event C.
- statement:

$$P(A, B \mid C) = P(A \mid C)P(B \mid C)$$

Conditional Independence in an example

- Charlie calls Alice and reads her script C, then she calls Bob and reads him the same
- A week later we ask Alice to repeat the story Charlie told her, she remembered A, a noisy version of C
- We ask Bob the same, he recounts B, a different noisy version of C
- Are A and B independent? No! $P(A, B) \neq P(A)P(B)$
  - If we learn A from Alice, we can get a good guess about B from Bob
- If we knew C, would hearing A give us more information about B?
  - No, because all the shared information between A and B is explained by C, so:



ABC

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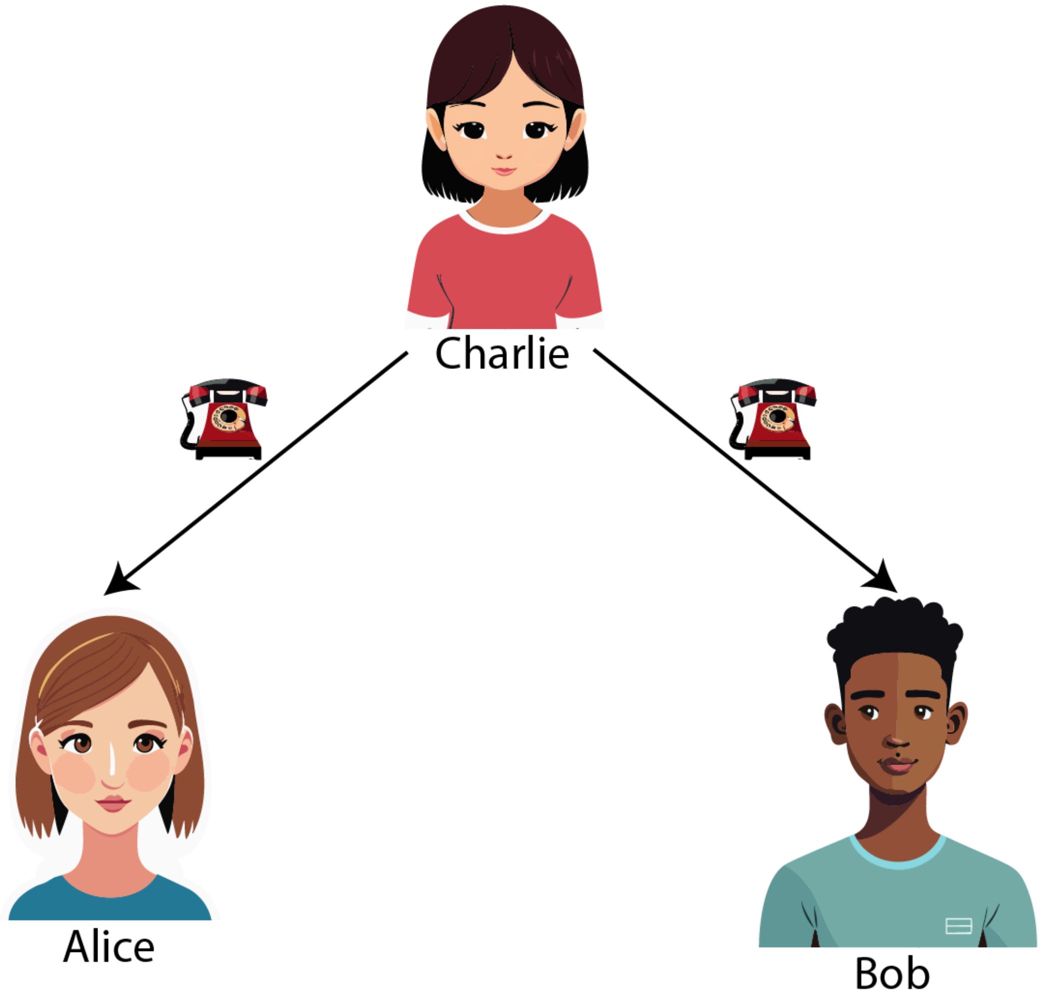

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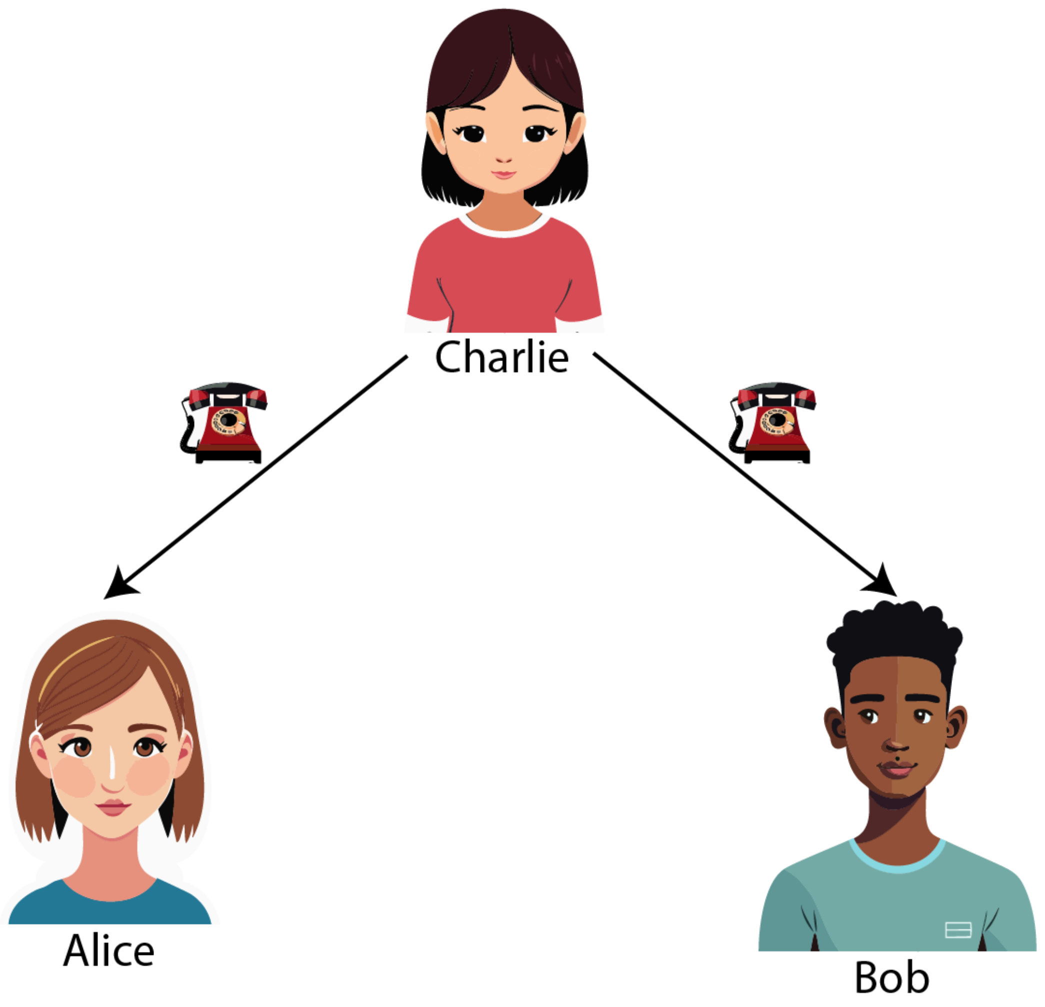

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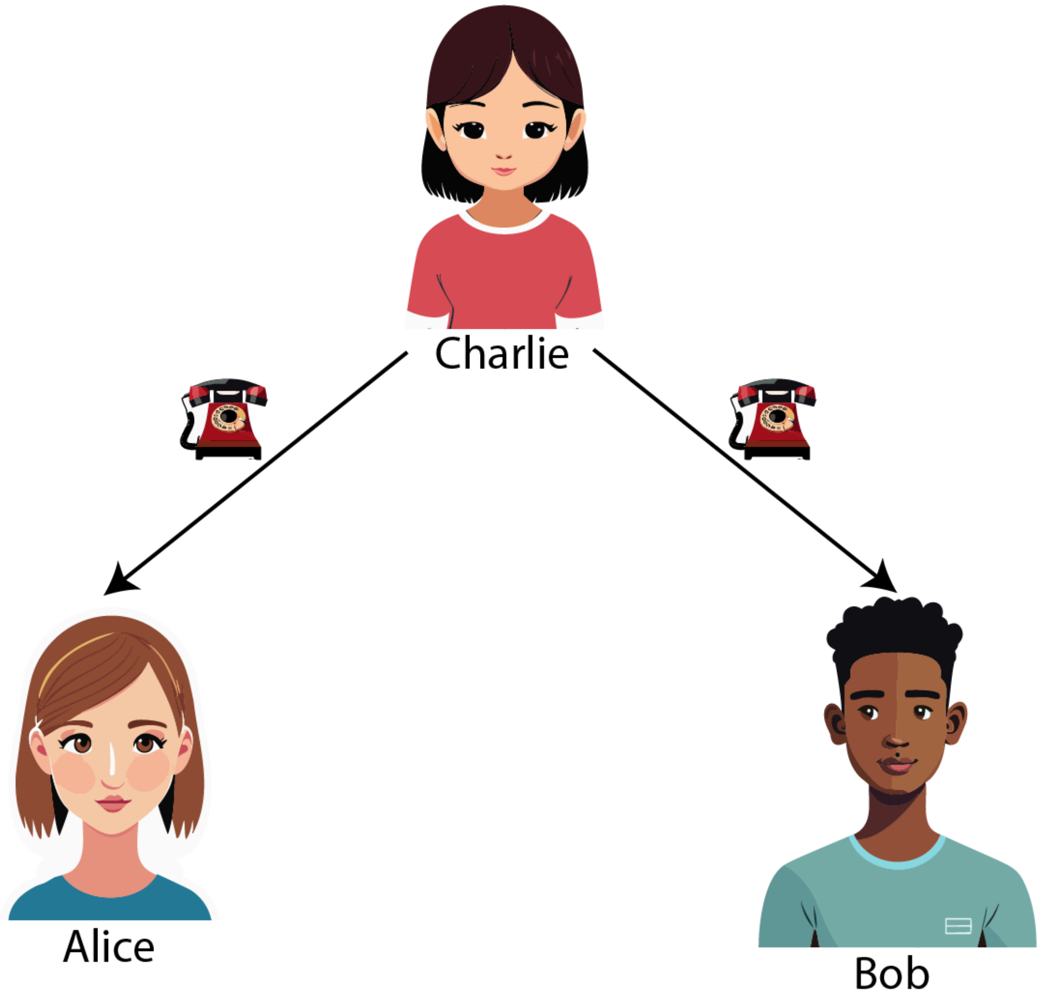
  - $P(A, B) \neq P(A)P(B)$
  - $P(A, B \mid C) = P(A \mid C)P(B \mid C)$
- Variables can be marginally dependent but conditionally independent (and vice-versa)



ABC

Conditional independence, stated differently

$$P(A|B,C) = P(A|C)$$

statement	interpretation
$P(A,B) = P(A)P(B)$	(marginal) independence of A and B
$P(A,B|C) = P(A|C)P(B|C)$	conditional independence of A and B given C
$P(A|B,C) = P(A|C)$	conditional independence of A and B given C

Conditional independence, stated differently

$$P(A \mid B, C) = P(A \mid C)$$

statement	interpretation
$P(A, B) = P(A)P(B)$	(marginal) independence of A and B
$P(A, B \mid C) = P(A \mid C)P(B \mid C)$	conditional independence of A and B given C
$P(A \mid B, C) = P(A \mid C)$	conditional independence of A and B given C

- both statements of conditional independence can be shown to be equivalent (when the involved conditional probabilities are well-defined) Section 5.2

Assumption parlance

Hierarchy of conditions / assumptions

- necessary assumption:
  - A **must** hold for B to be true
  - *having a heart is necessary for having a heart rate*
- sufficient assumption:
  - B is always true when A holds
  - *Being a square is sufficient to be a rectangle*
- strong assumption:
  - requires *strong* evidence, we'd rather not make these
- weak assumption:
  - requires *weak* evidence
- strong vs weak assumption are judged on relative terms
  - if assumption A is sufficient for B, B cannot be a stronger assumption than A

Proofs

Law of total (conditional) probability

We are asked to prove:

$$P(A \mid C) = \sum_b P(A \mid B = b, C)\, P(B = b \mid C)$$

$$\begin{aligned} P(A \mid C) &= \sum_b P(A, B = b \mid C) && \text{(sum rule)} \\ &= \sum_b P(A \mid B = b, C)\, P(B = b \mid C) && \text{(product rule)} \end{aligned}$$

Conditional independence equivalent statements

We will prove that the conditional independence statement

$$P(A \mid B, C) = P(A \mid C)$$

is **equivalent** to

$$P(A, B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

using basic rules of probability.

Conditional independence equivalent statements

✅ Proof (⇐ direction):

Assume

$$P(A, B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

By the product rule,

$$P(A, B \mid C) = P(A \mid B, C) \cdot P(B \mid C)$$

Comparing both expressions:

$$P(A \mid B, C) \cdot P(B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

Divide both sides by $P(B \mid C) > 0$, we get:

$$P(A \mid B, C) = P(A \mid C)$$

Conditional independence equivalent statements

✅ Proof (⇒ direction):

Assume

$$P(A \mid B, C) = P(A \mid C)$$

Again by the product rule:

$$P(A, B \mid C) = P(A \mid B, C) \cdot P(B \mid C)$$

Substitute $P(A \mid C)$ for $P(A \mid B, C)$, we get:

$$P(A, B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

Conditional independence equivalent statements

✅ Conclusion:

$$P(A \mid B, C) = P(A \mid C) \quad \iff \quad P(A, B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

as required.