# Structural Causal Models

introduction

Wouter van Amsterdam

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# DAG-recap

# In past lectures on DAGs

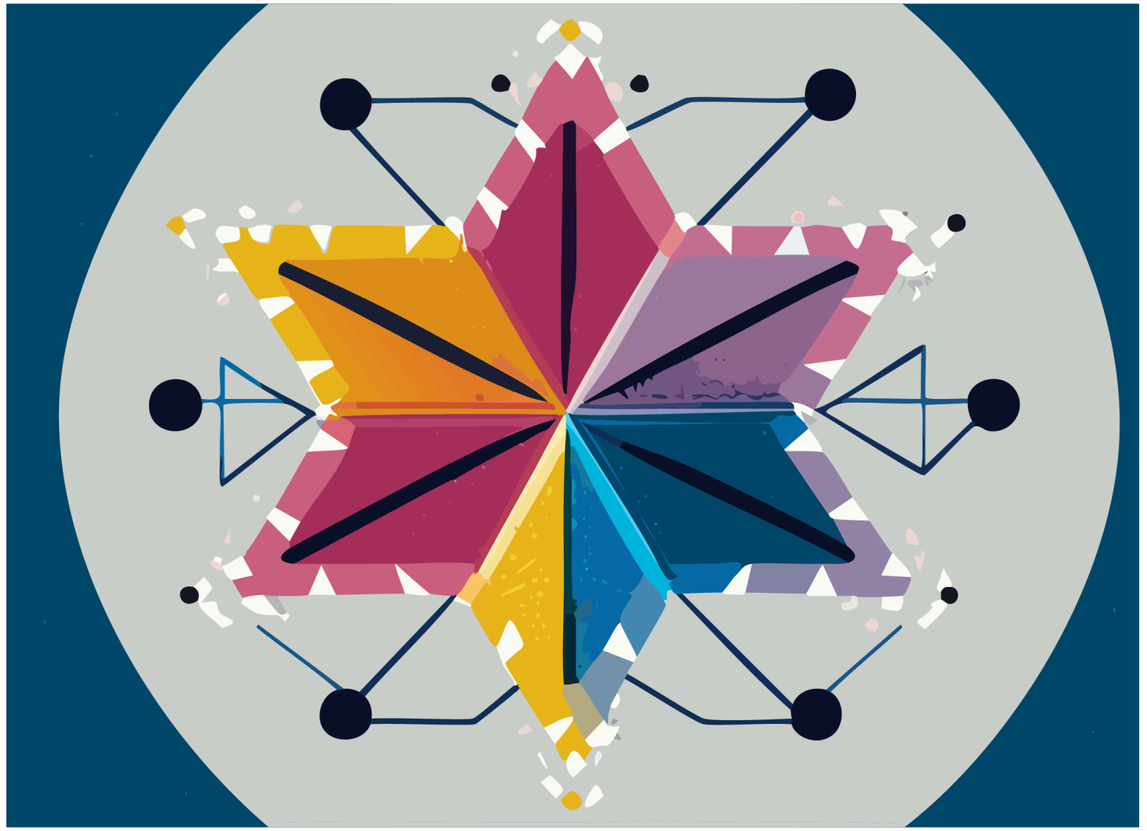
1. causal directed acyclic graphs (DAGs) encode assumptions on what variables cause what
2. an intervention is defined as a mutilation of this DAG where the treatment variable no longer ‘listens’ to its parents
3. a causal effect is the effect of an intervention
4. DAG patterns:
    - fork (confounding)
    - chain (mediation)
    - collider
5. typically:

- condition on confounders, don’t condition on mediators or colliders

6. in more complex DAGs, use d-separation to check identifyability
7. backdoor criterion

# In this lecture: structural causal models (SCMs)

$$
\begin{aligned}
U_Z, U_T, U_Y &\sim p(U) \\
Z &= f_Z(U_Z) \\
T &= f_T(Z, U_T) \\
Y &= f_Y(T, Z, U_Y)
\end{aligned}
$$



# Why SCMs?

- With DAGs we can:
  - express (non-parametric) prior knowledge
  - understand that seeing ≠ doing
  - know what variables to condition on for estimating treatment effect
- However,
  - DAGs and RCTs do not cover all causal questions
  - SCMs go a level deeper than DAGs
  - DAGs naturally ‘arise’ from SCMs
  - some questions are not identified when only specifying a DAG, but we may have additional information that can lead to identification
  - understand ‘identifyability’
  - SCM thinking aligns [^according to me] with physical thinking about the world and is a natural way to think about causality


WE NEED TO GO
DEEPER

# Topics of today

- SCMs: the world as computer programs
- interventions are submodels
- bonus queries:
  - counterfactuals
- Pearl Causal Hierarchy
- reflections on DAGs, limitations

# Structural Causal Models: definitions

# Think of the world as a computer program with a set of

- (endogenous) *variables*:
  - `surgery` = duration of surgery (hours)
  - `los` = length of stay in hospital post surgery (days)
  - `survival` = survival time (years)
- *background variables* (exogenous):
  - `u_surgery`, `u_los`, `u_survival`
- *functions* `f_` for each *variable* which depend on its *parents* `pa_` and its own *background* `u_`:
  - `surgery = f_surgery(pa_surgery,u_surgery)`
  - `los = f_los(pa_los, u_los)`
  - `survival = f_survival(pa_survival, u_survival)`

Together these define a *Structural Causal Model* (see definition 7.1.1 in Pearl 2009, and further) (notation: $M =< U, V, F >$)

# Structural Causal Model 1

```r
f_surgery <- function(u_surgery) { # pa_surgery = {}
  u_surgery
}
f_los <- function(surgery, u_los) { # pa_los = {surgery}
  surgery + u_los
}
f_survival <- function(surgery, los, u_survival) { # pa_survival = {sugery, los}
  survival = los - 2 * surgery + u_survival
}

scm1 <- function(u_surgery, u_los, u_survival) {
  surgery  = f_surgery(u_surgery)
  los      = f_los(surgery, u_los)
  survival = f_survival(surgery, los, u_survival)
  c(surgery=surgery, los=los, survival=survival)
}
scm1(2, 1, 5)
```

```
surgery      los survival
      2        3        4
```

# Recursive Structural Causal Models imply a Directed Acyclic Graph

An SCM is *recursive*, i.e. *acyclic* when following the chain of parents, you never end up at the same variable twice

```r
scm1 <- function(u_surgery, u_los, u_survival) {
  surgery  = f_surgery(u_surgery)
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  c(surgery=surgery, los=los, survival=survival)
}
```

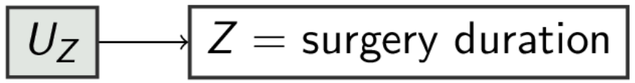

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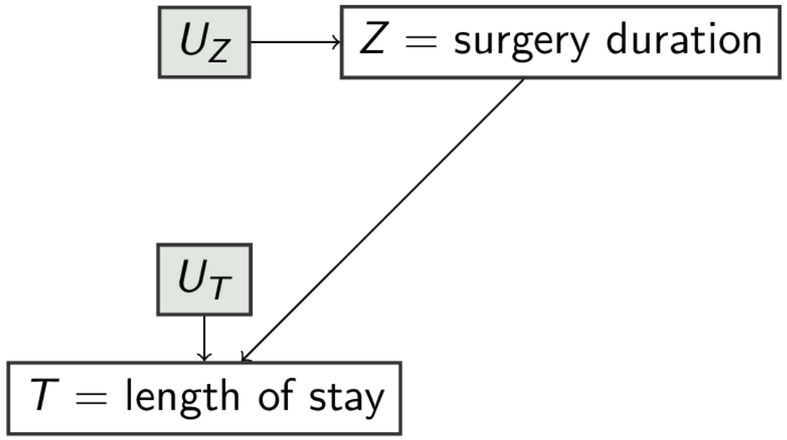
  c(surgery=surgery, los=los, survival=survival)
}
```



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  los      = f_los(surgery, u_los)
  survival = f_survival(surgery, los, u_survival)
  c(surgery=surgery, los=los, survival=survival)
}
```
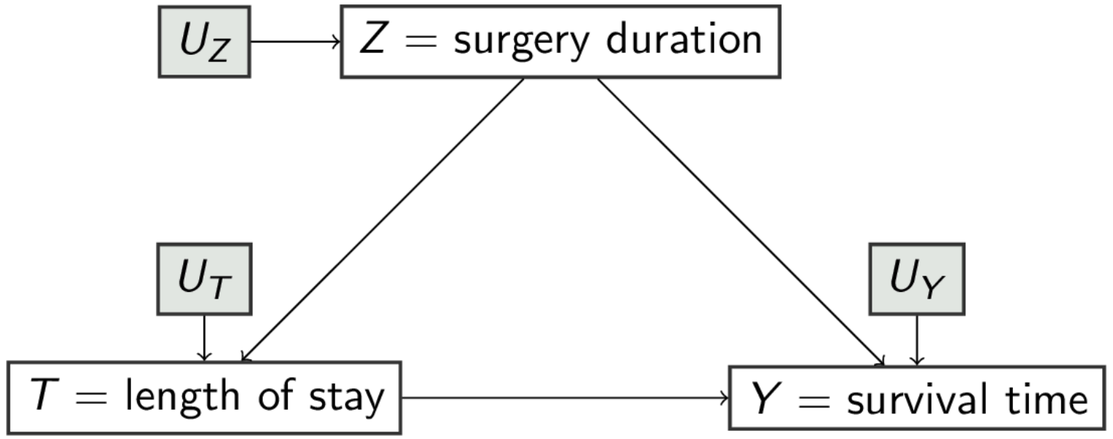




`scm1` (without specifying the `f_`s) and the DAG are equivalent (they describe the same knowledge of the world)

for the remainder, we assume recursiveness

# Submodel and Effect of Action

- **submodel**: in `scm1` replace `f_los` with a specific *value*, e.g. `7` days

```
submodel7 <- function(u_surgery, u_los, u_survival) {
  surgery = f_surgery(u_surgery)
  los = 7
  survival = f_survival(surgery, los, u_survival)
  c(surgery=surgery, los=los, survival=survival)
}

submodel7(2, 1, 5)
```

```
surgery      los survival
      2        7        8
```

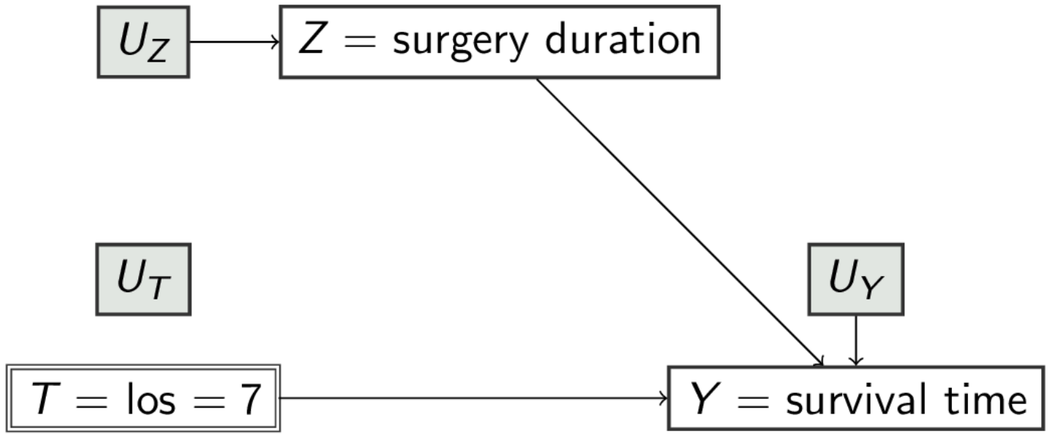
- **effect of action**: resulting SCM of submodel (notation: $M_x =< U, V, F_x >$)

# Submodel and Effect of Action as a mutilated DAG

In `scm1` replace `f_los` with a specific *value*, e.g. `7` days (notation: $M_x$)

```r
submodel7 <- function(u_surgery, u_los, u_survival) {
  surgery = f_surgery(u_surgery)
  los = 7
  survival = f_survival(surgery, los, u_survival)
  c(surgery=surgery, los=los, survival=survival)
}

submodel7(2, 1, 5)
```

```
surgery       los survival
      2         7        8
```




The DAG describes a submodel where $T$ no longer 'listens' to any variables but is controlled to be equal to a specific value (e.g. 7)

The *Effect of Action* $do(X = x)$ is defined as the submodel $M_x$.

# Specifying a distribution for exogenous variables U

- Exogenous variables U represent random variation in the world.
- We can specify a *distribution* for them (e.g. Gaussian, Uniform)

```r
sample_u <- function() {
    u_surgery  = runif(1,  2,  8)
    u_los      = runif(1, -1,  7)
    u_survival = runif(1,  8, 13)
    c(u_surgery=u_surgery, u_los=u_los, u_survival=u_survival)
}
sample_u()
```

```
u_surgery      u_los u_survival
 5.299693   4.308564  12.199266
```

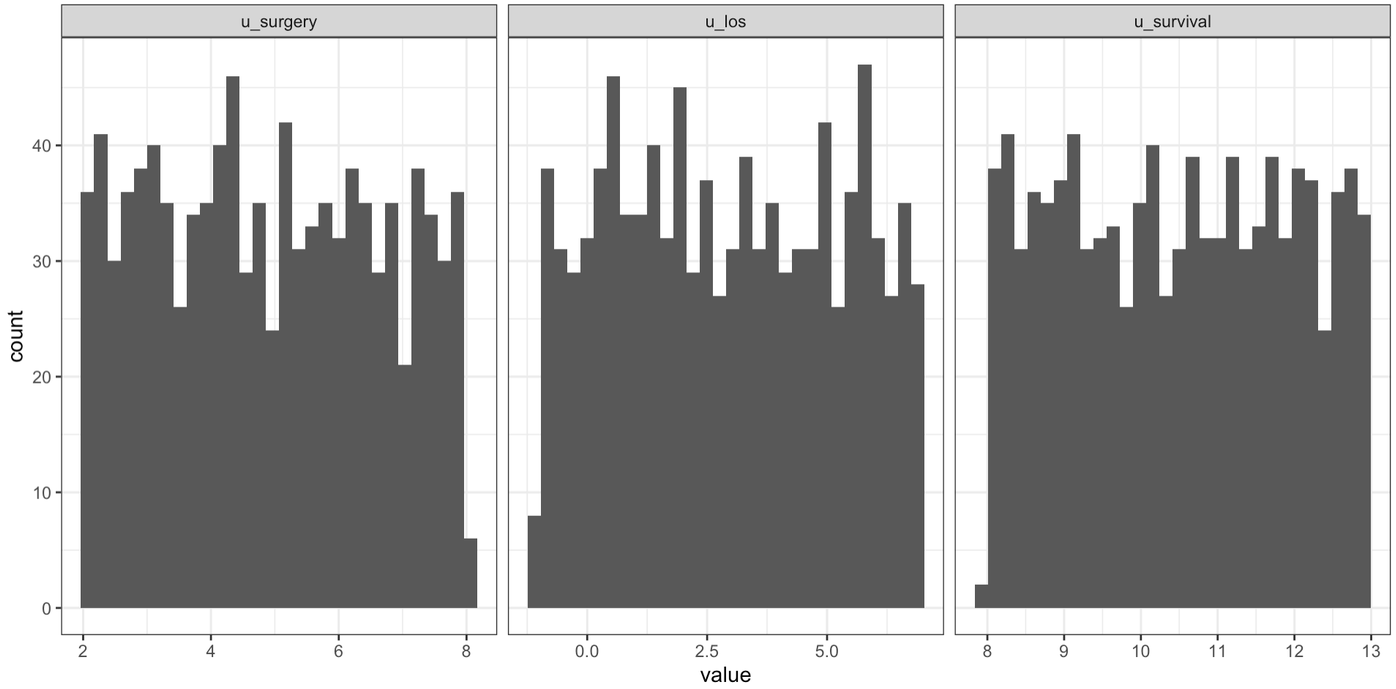



Figure 1: 1000 random samples of U

# A Probabilistic Causal Model is a SCM with a distribution over U

```r
sample_pcm <- function() {
  U <- sample_u()
  V <- scm1(U[['u_surgery']], U[['u_los']], U[['u_survival']])
  c(U, V)
}

sample_pcm()
```

```
u_surgery      u_los u_survival    surgery        los   survival
 3.019182   1.401587  10.914728   3.019182   4.420770   9.297133
```

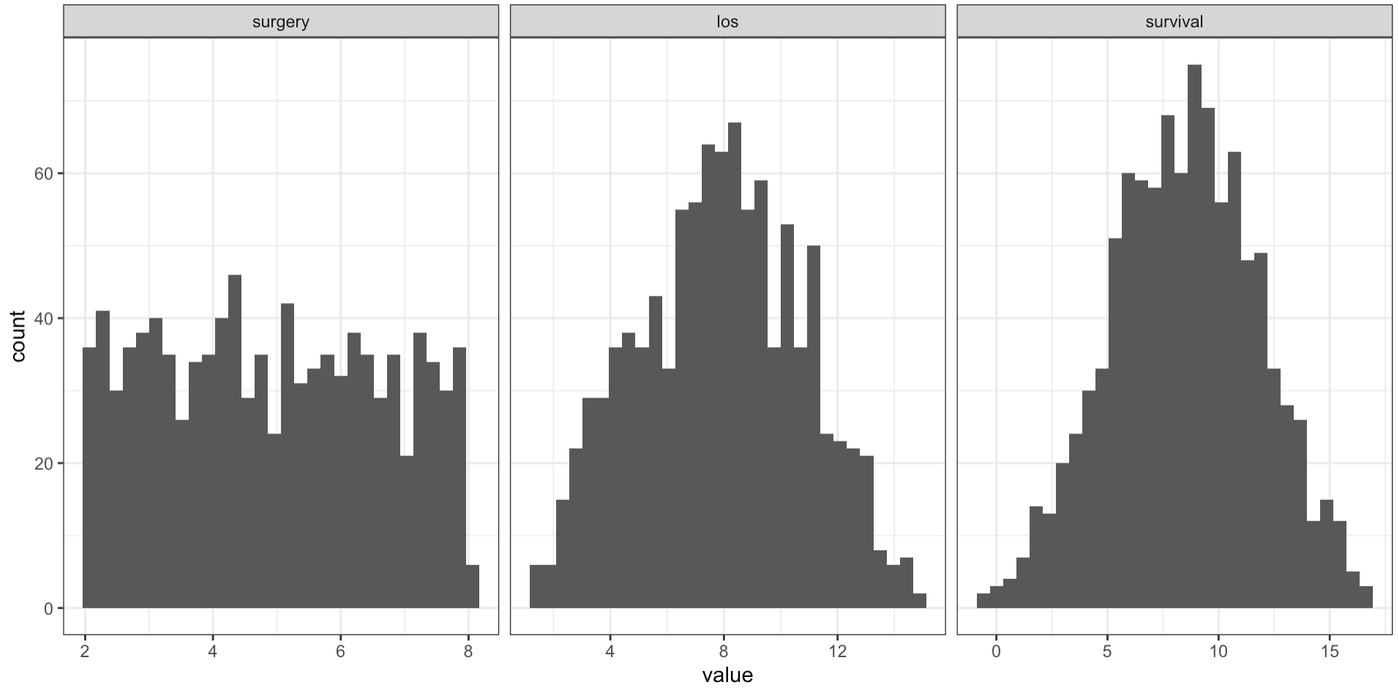



Figure 2: Realisations of endogenous variables V over random samples of U in Figure 1

# Calculating a treatment effect in a fully specified probabilistic causal model

- take random samples from `U`, push forward through `submodel7` and `submodel3`

```
# N = 1e3
# us <- map(1:N, ~sample_u())

v3s <- map(us, ~do.call(submodel3, as.list(.x)))
v7s <- map(us, ~do.call(submodel7, as.list(.x)))

v3df <- v3s |> map(~data.table(t(.x))) |> rbindlist()
v7df <- v7s |> map(~data.table(t(.x))) |> rbindlist()
v3df[, idx:=.I]
v7df[, idx:=.I]

dfa <- rbindlist(list(
  scm1=vdf,
  submodel3=v3df,
  submodel7=v7df
), idcol='model')

dfa[, list(mean_survival=mean(survival)), by="model"]
```

```
      model mean_survival
     <char>         <num>
1:     scm1      8.489856
2: submodel3      3.637293
3: submodel7      7.637293
```

# Identification

# Recap of definitions

- *Structural Causal model*:
  - endogenous variables $V$
  - exogenous (noise) variables $U$
  - deterministic functions `f_i(pa_i,u_i)`
- *Effect of Action* $\text{do}(T = t)$: *submodel* where `f_T` replaced with fixed value `t`
- *Probabilistic Causal Model*: SCM + distribution over U

# In the real world

- knowing the SCM is a super-power: you basically know everything revelant about the system, but in the real world:
- we do not observe $U$
- we typically do not know `f_`
  - we may be willing to place *assumptions* on `f_` (e.g. generalized linear models)
- we are presented with realizations $V_i$ of this SCM over a random sample of U
  - this is another assumption on the *sampling* but this is largely orthogonal to causal inference
- we may be interest in knowing:

  1. what is the expected survival time if we *always* admit patients for exactly 7 days?

When and how might we learn the answer to such questions?

# Identification

Causal effect identification:

## Definition 3.2.3 (Identifiability)

Let $Q(M)$ be any computable quantity of a model $M$.

We say that $Q$ is **identifiable** in a class $\mathbb{M}$ of models if, for any pairs of models $M_1$ and $M_2$ from $\mathbb{M}$,

$Q(M_1) = Q(M_2)$ whenever $P_{M_1}(y) = P_{M_2}(y)$.

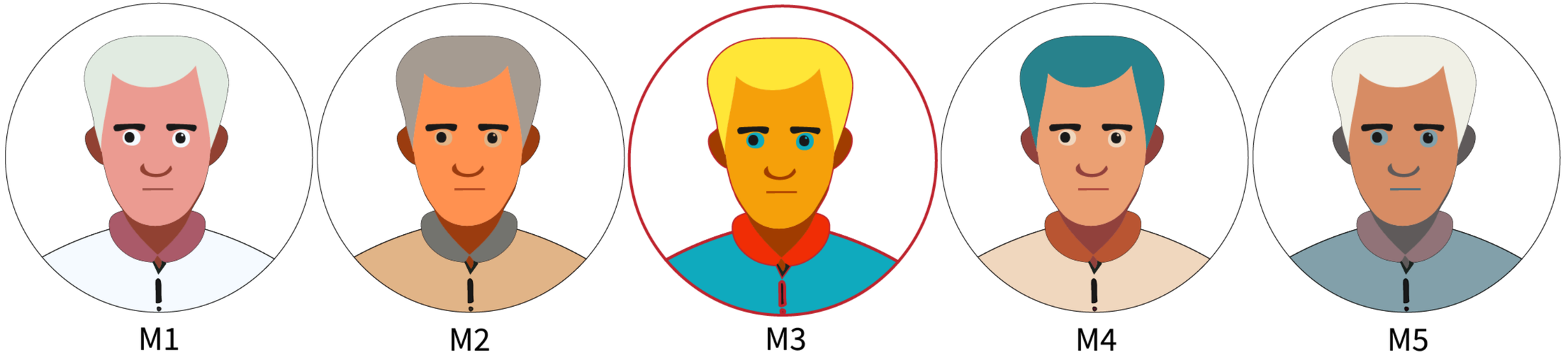
If our observations are limited and permit only a partial set $F_M$ of features (of $P_M(y)$) to be estimated,

we define $Q$ to be identifiable from $F_M$ if $Q(M_1) = Q(M_2)$ whenever $F_{M_1} = F_{M_2}$.

# Idenfitication in pictures

Someone killed the priest (†), we want to know who-dunnit (= $Q$)

Based on prior knowledge we have *5 suspects* (all the SCMs compatible with our DAG)




If we had full data, we would know it was $M_3$

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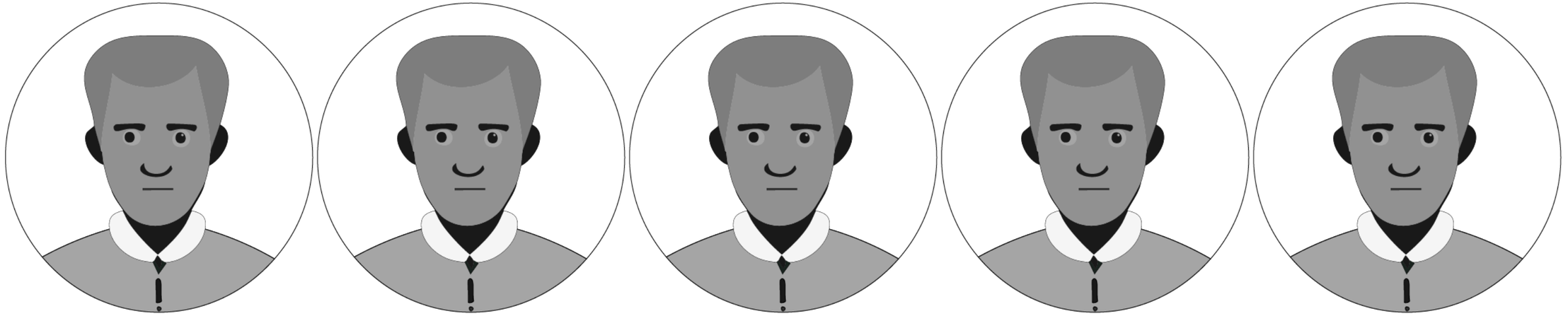
Based on prior knowledge on *5 suspects* (all the SCMs compatible with our DAG)



If we had full data, we would have know it was $M_3$

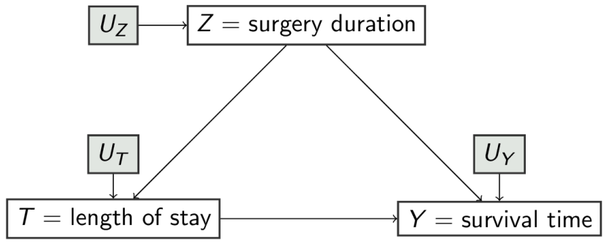
Unfortunately, it was dark an we only got a gray-scale image of the perpetrator

All our suspects (models) lead to the same partial observations

Based on *observed data* and *assumptions* we cannot *identify* the answer to our question $Q$,

i.e. multiple models with different answers for $Q$ fit the observed data equally well

# Not identified vs estimand




The backdoor adjustment in this DAG means the correct estimand is:

$$P(Y|\text{do}(T)) = \sum_z P(Y|T,z)P(Z=z)$$

- If we did not observe $Z$, we could still come up with a latent-variable model for $Z$ and a model for $Y|T,Z$ and get a value.
- However, we can formulate multiple distinct latent variable models that each yield a different treatment effect (i.e. the output of the estimand)
- But these latent variable models all fit the *observed* data equally well
- So we cannot identify the treatment effect

# Seeing is not doing

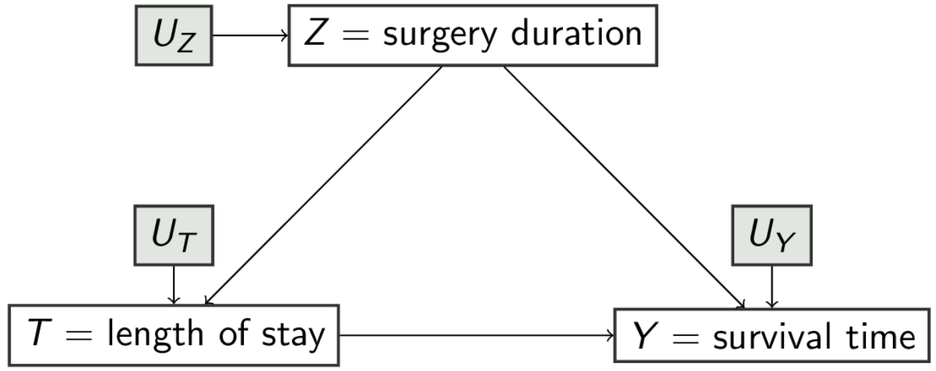
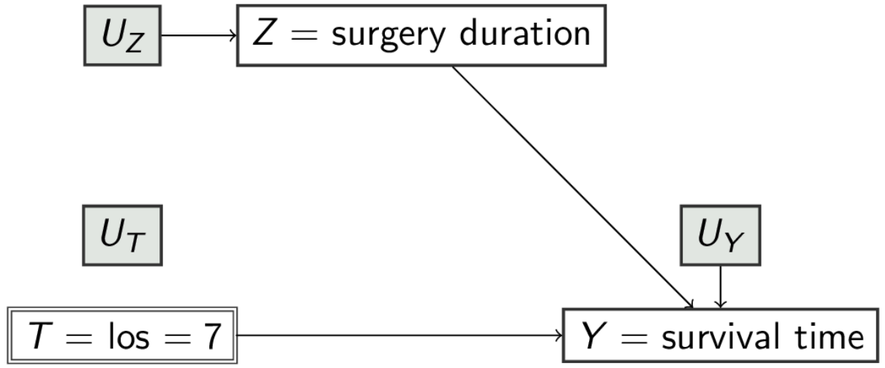



Figure 3:

$$P(Y|T) = \sum_z P(Y|T,z)P(Z=z|T)$$




$$P(Y|T) = \sum_z P(Y|T,z)P(Z=z|T)$$
$$=^2 \sum_z P(Y|T,z)P(Z=z)$$

Figure 4: $^2$ because in the intervened DAG, $Z$ is independent of $T$

- $P(Y|\text{do}(T)) \neq P(Y|T)$ is Pearl's definition of confounding (def 6.2.1)
- this shows why RCTs are special (i.e. no backdoor paths into $T$)

# Another path to identification: parametric assumptions

- for example:
  - assumption 1: $\mathbb{M}_1$, all SCMs with same DAG
  - assumption 2: $\mathbb{M}_2$ SCMs with linear functions and Gaussian error terms
  - assumption 1+2: $\mathbb{M} = \mathbb{M}_1 \cap \mathbb{M}_2$ (DAG + linear gaussian)
- many more effects are identified in this setting
- ‘works’ with unobserved confounding, positivity violations
- caveats:
  - much harder to determine identifyability (no analogue of backdoor-rule)
  - prefer weaker assumptions over stronger assumption

# Defining counterfactuals and the causal hierarchy (of questions)

# Counterfactuals

- all of the above can be achieved with DAGs, but we haven't used SCMs *unique power* yet: counterfactuals
- RCT / DAG questions: *What is the expected survival if we keep all patients in the hospital for 7 days?*

# Take it one level higher: counterfactuals

For patient Adam we had this data:

- surgery duration: 4 hours
- length of stay: 3 days
- survival: 4 years

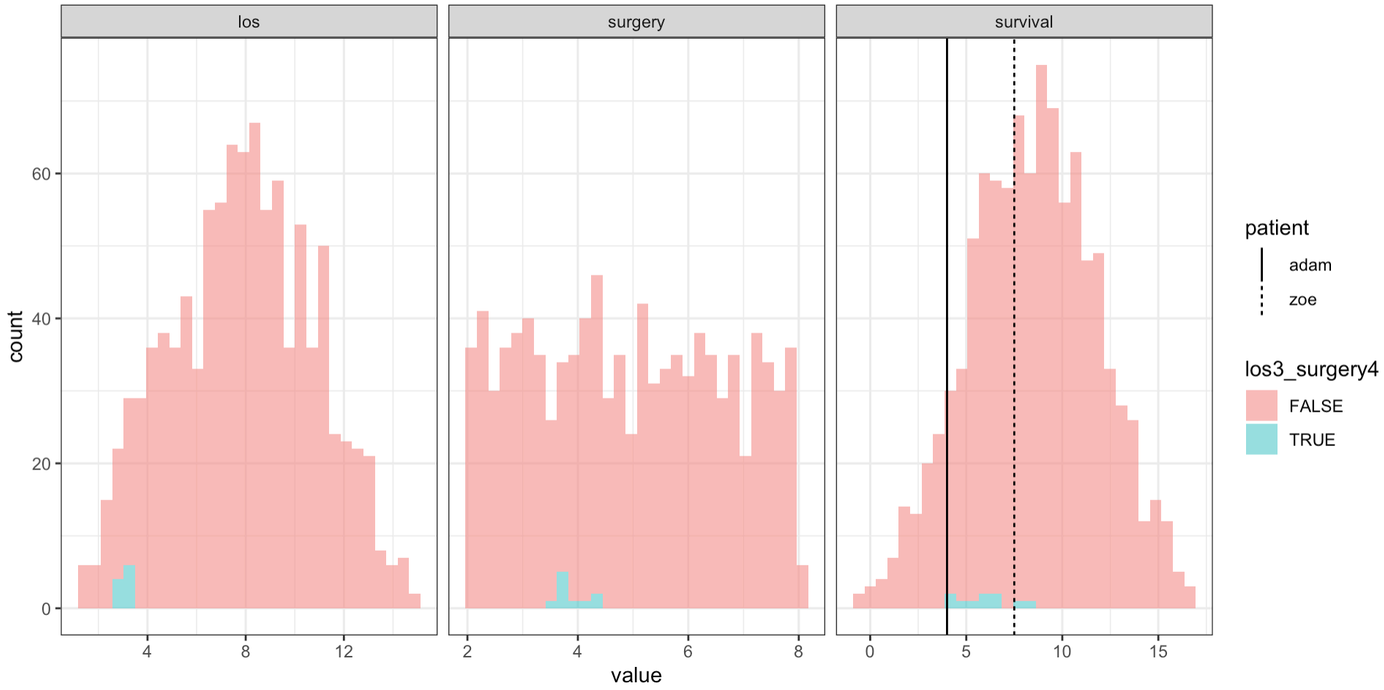
For patient Zoe we had this data:

- surgery duration: 4 hours
- length of stay: 3 days
- survival: 7.5 years

- we do not observe Adam's/Zoe's $U$
- What would the expected survival have been had Adam/Zoe been kept in the hospital for 7 days?

# Adam versus Zoe

- Average causal effects in subgroup with `surgery=4`:
  - 3-days LOS: 5.5
  - 7-days LOS: 9.5




- what do we expect for Adam and Zoe if they would have been kept in the hospital for 7 days?

# Computing counterfactuals with SCMs

- Given our information on the structural equation for `survival` (Section 2.2):

$$\text{survival} = \text{los} - 2 * \text{surgery} + u_{\text{survival}}$$

- and observed values on Adam's and Zoe's `surgery` AND `survival` following `los=3`
- we can compute their individual $u_{\text{survival}}$:

| patient | surgery | los | survival | u_survival | survival7 |
|---|---|---|---|---|---|
| Adam | 4 | 3 | 4 | 9 | 8 |
| Zoe | 4 | 3 | 7.5 | 12.5 | 11.5 |

- **and (counterfactual) survival under 7 days LOS**

# Computing counterfactuals

- notation: $P(Y_{t'} = y' \mid T = t, Y = y)$ where $Y_{t'}$ means "set $T = t'$ through intervention"
- steps:
  1. Abduction (update $P(U)$ from observed evidence)
  2. Action (modify the treatment)
  3. Prediction (calculate outcomes in submodel, putting in the updated $P(U)$)

# Pearl’s Causal Hierarchy (of questions)

If you have data to solve the upper, you can solve the lower ranks too (Bareinboim et al. 2022)

1. counterfactuals
2. interventions
3. associations

# Where do we get this knowledge from?

- not from observational data
- not from RCTs
- from assumptions
- can get bounds from combinations of RCT data and observational data
- caveat: some say the hierarchy is upside down because you go further away from data and closer to unverifiable assumptions the ‘higher’ you get

# Not covered but also possible:

- DAGs:
  - *soft intervention*: don’t set treatment to fixed value but replace function with other function of variables
  - express patterns for missing data by including missingness indicators
- SCMs:
  - probability of sufficiency
  - probability of necessity

# References

Bareinboim, Elias, Juan Correa, Duligur Ibeling, and Thomas Icard. 2022. “On Pearl’s Hierarchy and the Foundations of Causal Inference (1st Edition).” In *Probabilistic and Causal Inference: The Works of Judea Pearl*, edited by Hector Geffner, Rita Dechter, and Joseph Halpern, 507–56. ACM Books.

Pearl, Judea, ed. 2009. “The Logic of Structure-Based Counterfactuals.” In *Causality*, 2nd ed., 201–58. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9780511803161.009.